The influence of girth and degeneracy on Semitotal dominating set

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Abstract. In a graph G without an isolated vertex, a set $D \subseteq V(G)$ is called a semitotal dominating set, if D is a dominating set and for each vertex $v \in D$, there exists another vertex $u \in D$ such that distance between u and v is at most two in G. Given a graph G, the semitotal dominating set problem (SD) require computing a minimum cardinality semitotal dominating set of G. The SD problem is a relaxed version of the total dominating set problem, which is extensively studied in the literature. It is known that SD is NP-hard on chordal graphs, bipartite graphs, and planar graphs. On the other side it is polynomial-time solvable for interval graphs, strongly chordal graphs, convex bipartite graphs, distance-hereditary graphs, and AT-free graphs. In this paper, we begin exploring the parameterized complexity of the problem. We present an interesting dichotomy for SD from the parameterized complexity perspective when parameterized by the solution size: on graphs with girth three (four), the problem is W[2]-hard, and on graphs with girth at least five, the problem is fixed-parameter tractable. On d-degenerate graphs, we show that SD is W[2]-hard when parameterized by the solution size. Further, when parameterized by cliquewidth or treewidth of the input graphs, we establish that SD is fixed-parameter tractable.

Keywords: Semitotal dominating set · Parameterized complexity · Girth · Degeneracy · Treewidth

1 Introduction

The domination set problem (DS) is one of the well-studied problems in the field of graph theory [11,12,10]. A set $D \subseteq V(G)$ is a dominating set if each vertex $v \in V(G) \setminus D$ is adjacent to at least one vertex in D. Motivated by the various real-world applications, various constraints have been imposed on D, such as total domination, semitotal domination, connected domination, and outer connected domination. In this study, we revisit the semitotal dominating set problem which is a generalization of the total dominating set problem (TD).

The total dominating set problem is the most natural variant of the dominating set problem. A set $D \subseteq V(G)$ is a total dominating set if D is a dominating set and each vertex $v \in D$ has a neighbor in D. A survey of the computational complexity study of TD is reported in [13]. Goddard et al. [9] initiated the study of the semitotal dominating set problem (SD). For a connected graph G, a set $D \subseteq V(G)$ is a semitotal dominating set if D is a dominating set and each vertex $v \in D$ has another vertex in D which is at a distance within two. SD of G asks for a minimum cardinality semitotal dominating set. It is important to highlight that $\gamma(G) \leq \gamma_{t2}(G) \leq \gamma_t(G)$ where $\gamma(G)$ is the domination number of G, $\gamma_t(G)$ is the total domination number of G, and $\gamma_{t2}(G)$ is the semitotal domination number of G. On the computational complexity front, it is known from [9] that SD is NP-complete for general graphs.

Considering the practical importance of SD in the context of wireless networks, given the fact that SD is NP-complete, it is natural to explore other avenues, such as parameterized complexity theory, approximation algorithms, and exact exponential-time algorithms. In this paper, we initiate this line of study from the perspective of parameterized complexity [6]. For decision problems with input size n, and a parameter k, the objective is to design an algorithm with running time $f(k)n^{O(1)}$ where f is a function of k. Problems that admit such an algorithm are said to be fixed-parameter tractable (FPT). Similarly, problems that do not admit such an algorithm are said to be fixed parameter intractable. There is a hierarchy of intractable parameterized problem classes above FPT, the main ones are as follows: FPT $\subseteq$ W[1] $\subseteq$ W[2] $\subseteq \quad \ldots \quad \subseteq$ W[P] $\subseteq$ XP. In this paper, we study the parameterized complexity of SD with respect to both natural parameters (the solution size) and structural parameters (cliquewidth, and treewidth).

Related works

It is known that SD is NP-complete for split graphs, planar graphs, chordal bipartite graphs [15], and circle graphs [16]. On the positive side, SD is polynomial-time solvable on trees [9], AT-free graphs [16], interval graphs [15], strongly chordal graphs [19], and block graphs [14]. Approximation study of SD can be found in [16,15,18,8].

Although SD has been studied from the classical and approximation perspective, this study has not been taken into account from the perspective of parameterized complexity theory. In this paper, we initiate the parameterized complexity study of SD.

Our results. In this paper, we shall study the following problem in the realm of parameterized complexity.

The parameterized semitotal dominating set problem (PSD)
Input: A graph G, and an integer k.
Parameter: k

Question: Does there exist a set S of size at most k such that $V(G) \setminus S$ has a neighbor in S and every vertex in S has a vertex in S which is at a distance one or two.

Many dominating set variants such as DS, TD, and the connected dominating set problems have been analyzed in graphs with respect to girth. It is shown that many of these variants exhibit an FPT algorithm when parameterized by the solution size on graphs with girth at least five and is $W[2]$-hard for graphs with girth three or four. We explore a similar direction for PSD, and obtained the following results.

Theorem 1. PSD *is W[2]-hard on graphs with girth three when parameterized by the solution size.*

To prove the theorem, we give a parameterized reduction from the dominating set problem on general graphs to PSD on graphs with girth three.

Theorem 2. PSD *is W[2]-hard on graphs with girth four when parameterized by the solution size.*

We prove the above theorem for graphs with girth four by giving a parameterized reduction from the dominating set problem to PSD on C_3-free graphs. We observe that the reduction instances are bipartite graphs, in particular, the reduction instances have a cycle of length four. Thus, we have the following two corollaries.

Corollary 1. PSD *is W[2]-hard on bipartite graphs when parameterized by the solution size.*

Corollary 2. PSD *is W[2]-hard on C_3-free graphs when parameterized by the solution size.*

Having observed the complexity of PSD on graphs with girth three and girth four, it is an interesting direction to look into the complexity of the problem on graphs with girth at least five (G_5 graphs) when parameterized by the solution size.

Theorem 3. PSD *is FPT on graphs with girth at least five parameterized by the solution size.*

By using a branching strategy, we give an exponential kernel for the problem on G_5 graphs. Our approach is similar to the one presented in [17].

The study of PSD on graphs with respect to girth gives us an interesting dichotomy. We shall next consider the parameterized complexity study of PSD with respect to the degeneracy of the input graph.

Theorem 4. PSD *is W[2]-hard on d-degenerate graphs when parameterized by the solution size .*

We prove the theorem by a reduction from r-regular graphs to PSD on d-degenerate graphs. Further, by the construction, we observe the reduction instances are bipartite graphs. Thus, the following corollary is true.

Corollary 3. PSD *is W[2]-hard on d-degenerate bipartite graphs when parameterized by the solution size.*

The structural parameters such as treewidth, cliquewidth, vertex cover, and twin cover expand the boundary of FPT for PSD. We analyze the complexity of PSD with respect to well-known structural parameters such as treewidth and cliquewidth.

Theorem 5. PSD *admits an FPT algorithm, when parameterized by cliquewidth of the input graph.*

By giving a monadic second-order logic formula for PSD, we shall prove that PSD parameterized by cliquewidth is FPT. From [5], it is known that if a problem is known to be in FPT with respect to the parameter cliquewith then the problem is also known to be in FPT with respect to the parameter treewith. From this, we show that PSD parameterized by the treewidth is in FPT. Further, we also present an explicit algorithm for PSD parameterized by the treewidth of the input graph.

Theorem 6. PSD *admits an FPT algorithm, when parameterized by treewidth of the input graph.*

To prove the above theorem, we present a dynamic programming-based algorithm.

It is known that treewidth of a d-degenerate graph is at most d [2]. Thus, PSD on d-degenerate graphs admit FPT algorithm when parameterized by d.

Preliminaries. In this paper, we consider connected, undirected, unweighted, and simple graphs. For a graph G, $V(G)$ denotes the vertex set, and $E(G)$ represents the edge set. For a set $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced on the vertex set S. The open neighborhood of a vertex v is $N_G(v) = \{u \mid \{u, v\} \in E(G)\}$ and the closed neighborhood of v is $N_G[v] = \{v\} \cup N_G(v)$. The degree of vertex v is $d_G(v) = |N_G(v)|$. For a set $S \subseteq V(G)$, the closed neighborhood of S is $N_G[S] = \bigcup_{v \in S} N_G[v]$. By the girth of a graph G, we mean the length of the shortest cycle in G. We say that a graph is a G_i graph if the girth of the graph is at least i. A graph G is a d-degenerate graph if every subgraph of G contains a vertex of degree at most d. A graph is called r regular if the degree of each vertex in the graph is r. A graph G is a bipartite graph if $V(G)$ can be partitioned into $X \cup Y$ such that X, Y are disjoint independent sets. A vertex satisfies semitotal domination property means that it is in dominating set and has another vertex which is in dominating set and is at a distance of one or two.

2 W[2]-hardness on graphs with girth three and four

We prove Theorem 1, by reduction from the dominating set problem (PDS). It is well known that PDS parameterized by solution size is W[2]-hard [17].

The dominating set problem (PDS)
Input: A graph G, and an integer k.
Parameter: k
Question: Does there exist a set S of size at most k such that $N_G[S] = V(G)$?

Proof (proof of Theorem 1). Given an instance (G, k) of PDS , we construct the corresponding instance of (H, k) in polynomial time. The vertex set of H is: $V(H) = X \cup Y \cup \{w\}$, let $X = \{x_i \mid v_i \in V(G)\}$, and let $Y = \{y_i \mid v_i \in V(G)\}$. The edge set of H is: $E(H) = \{\{x_i, y_j\}, \{y_i, x_j\}, \{y_i, y_j\} \mid \{v_i, v_j\} \in E(G)\} \cup \{\{x_i, y_i\} \mid 1 \leq i \leq n\} \cup \{\{w, y_i\} \cup 1 \leq i \leq n\}$.

The girth of the reduced instance H is three because the presence of a single edge in G creates a cycle of length three in H.

($\Rightarrow$) If G has a dominating set S of size at most k, then H has a semitotal dominating set of size at most k. Let $D = \{y_i \mid v_i \in S\}$. By our construction, any vertex in Y is at a distance two from each other. Thus D is a semitotal dominating set.

($\Leftarrow$) If H has a semitotal dominating set D of size at most k, then G has a dominating set S of size at most k. Suppose that $D \cap \{w\} \neq \emptyset$. If $D \cap Y \neq \emptyset$, then $D' = D \setminus \{w\}$. Further, if $D \cap Y = \emptyset$, then $D' = (D \setminus \{w\}) \cup \{y_i\}$, for some i. Let $S = \{v_i \mid v_i \in V(G),\ \{x_i, y_i\} \cap D' \neq \emptyset\}$. Since $|D'| \leq k$, $|S| \leq k$. Now we need to show that S is a dominating set of G. Suppose that there exists a vertex $v_i \in V(G)$ such that $N_G[v_i] \cap S = \emptyset$. Then it must be the case that $(N_H[x_i] \cup N_H[y_i]) \cap D = \emptyset$, which is a contradiction that D is a semitotal dominating set. Thus, it follows that PSD is W[2]-hard on graphs with girth three. □

*Proof (*Proof of Theorem 2*).* Given an instance (G, k) of PDS, we construct a corresponding instance of $(H, k+1)$ in polynomial time such that H is a bipartite graph. The vertex set of H is: $V(H) = X \cup Y \cup \{w, z\}$, let $X = \{x_i \mid v_i \in V(G)\}$, and let $Y = \{y_i \mid v_i \in V(G)\}$. The edge set of H is: $E(G) = \{\{x_i, y_j\}, \{y_i, x_j\} \mid \{v_i, v_j\} \in E(G)\} \cup \{\{x_i, y_i\} \mid 1 \leq i \leq n\} \cup \{\{w, y_i\} \cup 1 \leq i \leq n\} \cup \{\{w, z\}\}$.

The girth of the reduced instance H is four because H is bipartite, and the presence of a single edge in G creates a cycle of length four.

($\Rightarrow$) If G has a dominating set S of size at most k, then H has a semitotal dominating set of size at most $k + 1$. Let $D = \{y_i \mid v_i \in S\} \cup \{w\}$. By our construction, any vertex in Y is at a distance two from each other. Thus D is a semitotal dominating set.

($\Leftarrow$) If H has a semitotal dominating set D of size at most $k + 1$, then G has a dominating S of size at most k. Observe that $D \cap \{w, z\} \neq \emptyset$. Let $S = \{v_i \mid v_i \in V(G),\ \{x_i, y_i\} \cap D \neq \emptyset\}$. Since $|D| \leq k + 1$, $|S| \leq k$. Now we need to show that S is a dominating set of G. Suppose that there exists a vertex $v_i \in V(G)$ such that $N_G[v_i] \cap S = \emptyset$. Then it must be the case that $(N_H[x_i] \cup N_H[y_i]) \cap D = \emptyset$, which is a contradiction that D is a semitotal dominating set. Thus, it follows that PSD is W[2]-hard on graphs with girth four. □

3 FPT on G_5 graphs (graphs with girth at least five)

In this section we give a kernelization algorithm for PSD on graphs having girth at least 5. We begin with the following result which holds for graphs of girth at least 5.

Lemma 1. *If G is a graph of girth at least 5 then for any vertex $v \in V(G)$, $G[N_G(v)]$ is an independent set. Moreover, no two vertices in $N_G(v)$ has a common neighbor other than v.*

The idea we adopted here is similar to the idea of computing a k-sized dominating set in graphs that exclude cycles of length three and four (girth at least 5) [17]. In our algorithm we first identify a set of vertices of a graph G that must be included in any semitotal dominating set of size at most k.

Lemma 2. *Let G be a graph of girth at least 5. If a vertex $v \in V(G)$ has degree more than k in G then v belongs to each semitotal dominating set of G.*

Proof. Let G be a graph of girth at least 5 and D be a semitotal dominating set of G such that $|D| \leq k$. Moreover, let $v \in V(G)$ be a vertex of degree more than k and $v \notin D$. From Lemma 1, we note that $N_G(v)$ is an independent set and no two vertices in $N_G(v)$ has a common neighbour. This implies that, if $v \notin D$ then D contains at least $|N_G(v)|$ vertices to dominate vertices in $N_G(v)$, a contradiction, as $|D| \leq k$. □

Sketch of the algorithm: We use coloring scheme to keep track of vertices that satisfy semitotal domination property (orange), vertices that do not satisfy semitotal domination property (red), vertices that are dominated (white), and vertices that are not yet dominated (black). As the algorithm progresses the color of a vertex may change. Finally, either red and black becomes empty, or the number of vertices in the graph becomes bounded. In the former case we output the vertices that are colored orange as the semitotal dominating set, and for the latter case, we show that kernel of size $2^{k+k^2} + k^2 + k$.

In our algorithm we use colors to identify vertices of the input graph. Given a graph G of girth at least 5, we obtained a colored graph, call it orgb-graph G', whose vertices are colored with four colors – orange, red, green, and black. In particular, we use "red" color to identify the vertices which must be included in any semitotal dominating set of size at most k. Further, we use color "white" to identify vertices which have at least one red neighbour color "black" to identify vertices which do not have red neighbour. Moreover we color a vertex u with color "orange" if there is a vertex v in D such that $1 \leq d_G(u, v) \leq 2$.

Now, for this orgb-graph, we define an orgb-SD set to be a set $D \subseteq V$ that contains set of red and orange colored vertices and dominates all the black vertices. Given a orgb-graph and an integer k, we define the ORGB-SD problem which aims to compute an orbg-set of G of size at most k. Next, we prove the following observation.

Observation 1. *Let G be a graph and G' be a graph that is obtained form G by coloring all vertices of G with black color. Then G has a semitotal dominating set of size at most k if and only of G' has a orgb-SD set of size at most k.*

We construct D in incremental fashion following the coloring scheme which is defined below. The coloring scheme maintains four sets namely orange, red, white and black, which are updated progressively. As we progress through the algorithm, a vertex from one color set can move to another color set.

Note that initially all vertices are colored black, and the sets orange, red, and white are empty. As we progress through the algorithm, we change the color of a vertex as per the reduction rules, accordingly the sets are updated. Finally, we output orange as the solution.

Let G_i be an orwb-graph at an iteration i satisfying the following conditions.

1. Every white vertex is a neighbor of an orange or red vertex.
2. Every black vertex has no neighbor which is colored orange or red.
3. $|O \cup R| \leq k$

We shall now present the structural lemmata using which we show that the coloring scheme indeed works, and then we obtain an exponential kernel.

The following lemma holds true for semitotal dominating set on G_5 graphs.

Lemma 3. *Let G which is an orwb-graph be an instance of the* ORWB-SD *problem with a positive integer k as the parameter. Let v be the black or white vertex with more than $k - |O \cup R|$ black neighbors. Then, if G has a set of size at most $k - |O \cup R|$ that dominates all black vertices, then v must be part of every such set.*

Proof. The proof follows from Lemma 1 and Lemma 2.

We now present a set of rules using which the color of set of vertices changes as we progress through the algorithm. Using Lemma 3, we observe the following reduction rules using which we obtain an exponential kernel.

R1 If there is a white or black vertex v having more than $k - |O \cup R|$ black neighbors, then either v is colored orange if v has a distance within two neighbor in D or v is colored red if v has no distance within two neighbor in D.

R2 If there exists $u, v \in W$ such that $N_{G[O \cup R \cup B]}[u] = N_{G[O \cup R \cup B]}[v]$, then $W = W \setminus \{v\}$.

R3 If $|O \cup R| \geq k$, then stop and return NO.

Now we show that the reduction rules described above are safe. Let (G, k) be an instance of the problem and (G', k) be a reduced instance. Let (G, k) be a Yes-instance of the problem. As claimed in Lemma 3, any white or black vertex having more than $k - |O \cup R|$ black neighbors must belong to every semitotal dominating set of size at most k. Let $v \in V(G)$ that has more than $k - |O \cup R|$ black neighbors in G.

Let D be an orgb-SD set of size at most k. Then D contains all red and orange vertices of G and due of Lemma 3, $v \in D$. Thus, we note that the set D is a orgb-SD set of size at most k. Conversely, suppose D' is a orgb-SD set G'. Note that in our reduction, we are only coloring all the neighbours of v to be white. Other black vertices remains the same in G and G'. Moreover as G contains all the red and orange vertices of G', thus $v \in D'$. This implies that the set D' is an orgb-SD set of G of size at most k.

We apply the rules iteratively, until it is no longer possible. At iteration i, let the graph obtained be G_i with color classes O_i, R_i, W_i, and B_i.

Lemma 4. *Let G be an instance of the* ORWB-SD *problem, and let G_1 be the instance after applying rules R1 to R3 once. Let k be the integer parameter. Then G has a set of size at most $k - |R \cup O|$ dominating all vertices in B and satisfying the constraint that $|R| = \emptyset$ if and only if G_1 has a set of size at most $k - |R_1|$ dominating all vertices in B_1 and satisfying the constraint that $|R_1| = \emptyset$.*

From Lemma 3, and Lemma 4, we shall prove the following lemma.

Lemma 5. *Let (G, k) be a yes instance of the* ORWB-SD *problem and (G', k') be the reduced instance of (G, k) after applying rules **R1-R3** until no longer possible. Then the number of vertices in G' is $O(k^3 + 2^k + k + k^2)$.*

Proof Sketch: Observe that the number of white can be bounded with respect to their neighborhood as $k^3 + 2^k$, and the number of black vertices are bounded by at most k^2. Thus, the number of vertices in G' is $O(k^3+2^k+k+k^2)$.

Thus, the ORWB-SD problem can be solved in time $O((k^3 + 2^k + k + k^2)^k n^{O(1)})$ time. Since the resultant graph after applying reduction rules R1-R3 recursively, until no longer possible results in a graph with at most $k^3+2^k+k+k^2$ vertices, we can just try all possible subsets of size at most k for $k^3+2^k+k+k^2$ vertices.

4 PSD on d-degenerate graphs

4.1 W[2]-hardness on d-degenerate graphs

It is known from [3] that DS on r-regular graphs is W[2]-hard and this can be reduced in polynomial time to PSD in d-degenerate graphs using the following reduction algorithm.

Proof. (*proof of Theorem 4*) We map an instance (G, k) of DS on r-regular graphs to the corresponding instance (G', k) of PSD as follows: $V(G') = W \cup X \cup Y \cup Z$,
$W = \{w_i \mid v_i \in V(G)\}$, $X = \{x_i \mid v_i \in V(G)\}$,
$Y = \{y_i \mid v_i \in V(G)\}$, $Z = \{z_i \mid v_i \in V(G)\}$.
We shall now describe the edges of G',
$E(G') = E_1 \cup E_2$,
$E_1 = \{\{w_i, x_j\}, \{x_i, w_j\}, \{y_i, z_j\}, \{z_i, y_j\}, \{x_i, y_j\}, \{x_j, y_i\} \mid \{v_i, v_j\} \in E(G)\}$, and $E_2 = \{\{w_i, x_i\}, \{x_i, y_i\}, \{y_i, z_i\} \mid 1 \leq i \leq n\}$.

Observe that the maximum degree of G' is $2r+2$. Thus, G' is a $(2r+2)$-degenerate graph.

($\Rightarrow$) If G has a dominating set S of size at most k, then G' has a semitotal dominating set D of size at most $2k$. Let $D = \{y_i \mid v_i \in S\} \cup \{x_i \mid v_i \in S\}$. By our construction, for each $x_i, y_i \in D$, $\{x_i, y_i\} \in E(G')$. Thus D is a semitotal dominating set.

($\Leftarrow$) If G' has a semitotal dominating set D of size at most $2k$, then G has a dominating set S of size at most k. Suppose that $|(W \cup X) \cap D| \geq k$. Then $|(Y \cup Z) \cap D| \leq k$. Let $S = \{v_i \mid y_i \in D \text{ or } z_i \in D,\ 1 \leq i \leq n\}$. Now we need to show that S is a dominating set of G. Suppose that there exist a vertex $v_i \in V(G)$ such that $N_G[v_i] \cap S = \emptyset$. Then it must be the case that $(N_{G'}[x_i] \cup N_{G'}[y_i]) \cap D = \emptyset$, which is a contradiction that D is a semitotal dominating set. Thus it follows that PSD W[2]-hard on d-degenerate graphs. □

Observe that for d-degenerate graphs PSD is FPT when parameterized by d, which is established by Section 5.2.

5 Structural parameters

5.1 Parameterizing PSD by cliquewidth

Theorem 7. *[4,7] Let G be a graph, and let $\mathcal{P}$ be the graph property of G expressible in MSO_1. If cliquewidth of $G \leq k$, then $\mathcal{P}$ can be verified in time $O(f(k)n^{O(1)})$.*

Using Theorem 7, we shall now present a proof of Theorem 5.

Proof. (Proof of Theorem 5) Let G be a graph of bounded cliquewidth ($cw(G) \leq k$). PSD can be expressed in MSO_1 as follows;

$$\phi = \min(X) : \forall v \in V(G) \exists x \in X : (adj(v,x) \vee (v = x \wedge (\exists a \in X\ a \neq x \wedge adj(a,x))$$

$$\vee (\exists a \exists b\ a \neq b \neq x \wedge a \in X \wedge adj(b,x) \wedge adj(b,a))))$$

Hence the theorem. □

5.2 Parameterizing PSD by treewidth

The goal is to provide a dynamic programming algorithm on a tree that determines the minimum possible size of a dominating set of the input graph G. Let $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ be a nice tree decomposition of G which has treewidth at most l.

To prove the parameterized complexity of the problem parameterized by treewidth, we use the nice tree decomposition with the following nodes as defined in [6]:

1. Leaf node.
2. Introduce vertex node.

3. Introduce edge node.
4. Forget node.
5. Join node.

At any node $t \in \mathcal{T}$, $X_t \subseteq V(G)$ is the set of vertices present in the bag t. With each node t of the tree decomposition, we associate a subgraph G_t of G defined as follows:

$$G_t = (V_t, E_t = \{e \ : \ e \text{ is introduced in the subtree rooted at t}\}).$$

Our construction is similar to the one presented in [1]. Since we need to distinguish between the vertices in the dominating set and those that do not have a distance within two vertex in the dominating set, we use three colors. For the remaining vertices, we use two colors to differentiate whether or not they have been dominated by at least one vertex in the dominating set. Thus, a coloring of bag t is a mapping $f : X_t \to \{1, 1', 2, 0, 0'\}$.

- 1, meaning that the vertex is in the dominating set and satisfies semitotal dominating set property or has a neighbor in 2.
- $1'$, meaning that the vertex is in the dominating set and does not satisfy semitotal dominating set property.
- 2, meaning that the vertex is already dominated, not in D, and it is adjacent to a vertex in 1. We call vertices colored 2 as helper vertices as they are useful for some vertices in 1 to satisfy the semitotal dominating set.
- 0, meaning that the vertex is not in the dominating set and is already dominated
- $0'$, meaning that the vertex is not in the dominating set and is yet to be dominated.

Note that $f^{-1}(a)$ is a preimage of a in f, if exists, and $f^{-1}(a) = \emptyset$, otherwise.

Observe that at a node t, there are at most 5^l colorings of X_t. For a coloring f of X_t, let $c[t, f]$ denotes the minimum size of a semitotal dominating set D of V_t such that $D \cap X_t = f^{-1}(1) \cup f^{-1}(1')$, and every vertex in $f^{-1}(2) \cup f^{-1}(0)$ is adjacent to at least one vertex in $f^{-1}(1) \cup f^{-1}(1')$. Further every vertex in $f^{-1}(1')$ does not have a neighbor in $f^{-1}(2)$ or $f^{-1}(1)$. We call such a set D a minimum compatible set for t and f. We have $c[t, f] = +\infty$, if t and f are not minimum compatible in D.

At any node t, the value of $c[t, f]$ stores how many vertices are needed for a minimum semitotal dominating set of the graph that is visited upon till t.

By the property of nice tree decomposition $\mathcal{T}$, we know that for root node r of t, $X_r = \emptyset$ and $G_r = G$. Thus the root node has only one possible coloring which is empty coloring. Thus the size of the minimum semitotal dominating set in G is exactly the value of $c[r, \emptyset]$.

For a coloring f of X, $Y \subseteq X$, $f|_Y$ denotes the mapping of f restricted to Y. Next, we shall define the recursive formulas for the values of c at each node of the tree decomposition.

1. **Leaf node.** For a leaf node $X_t = \emptyset$. Hence there is only one coloring possible.

$$c[t, \emptyset] = 0$$

2. **Introduce vertex node.** Let t be an introduce node with a child t'. Then $X_t = X_{t'} \cup \{v\}$ for some $v \notin X_{t'}$. Observe that no edges are introduced to v and v is an isolated vertex in G_t. Hence, the isolated vertex v in G_t cannot receive color 2 or 0. Thus, we have the following values for $c[t, f]$.

$$c[t,f] = \begin{cases} +\infty, & \text{if } f(v) = 0 \text{ or } f(v) = 2 \\ c[t', f|_{X_{t'}}], & \text{if } f(v) = 0' \\ 1 + c[t', f|_{X_{t'}}], & \text{if } f(v) = 1 \text{ or } f(v) = 1' \end{cases}$$

   When a vertex v is introduced at t, v is an isolated vertex in the graph G_t. Thus v cannot be assigned 0 or 2. Hence $c[t, f] = +\infty$ when $f(v) = 0$ or $f(v) = 2$.
   Consider the case when $f(v) = 0'$, which means v is not yet dominated. Any coloring f is valid if and only if $f_{X_{t'}}$ is valid. Thus, we have $c[t, f] = c[t', f|_{X_{t'}}]$.

3. **Introduce edge node.** Let t be an introduce edge node labelled with an edge $\{u, v\}$. Let t' be the child of t. Then $V(G_t) = V(G_{t'})$ and $E(G_t) = E(G_{t'}) \cup \{\{u, v\}\}$. The edge $\{u, v\}$ is useful in the following ways; if $f(u) = 1$ or $f(u) = 1'$ and $f(v) = 0$ or $f(v) = 0'$, then taking precomputed solution for t' can help the color of v from $0'$ to 0 (or vice versa).

$$c[t,f] = \begin{cases} c[t', f_{v\to 0'}], & \text{if } f(u) \in \{1, 1'\} \text{ and } f(v) = 0 \\ c[t', f_{u\to 0'}], & \text{if } f(u) = 0 \text{ and } f(v) \in \{1, 1'\} \\ c[t', f_{u\to 1', v\to 1'}], & \text{if } f(u) = 1 \text{ and } f(v) = 1 \\ c[t', f_{u\to 1', v\to 0'}], & \text{if } f(u) = 1 \text{ and } f(v) = 2 \\ c[t', f_{u\to 0', v\to 1'}], & \text{if } f(u) = 2 \text{ and } f(v) = 1 \\ c[t', f_{u\to 1', v\to 2}], & \text{if } f(u) = 1 \text{ and } f(v) = 0 \\ c[t', f_{u\to 2, v\to 1'}], & \text{if } f(u) = 0 \text{ and } f(v) = 1 \\ c[t', f], & \text{otherwise} \end{cases}$$

   Note that any vertex in $f^{-1}(1)$ must either satisfy the semitotal dominating set property or must be adjacent to a vertex in $f^{-1}(2)$. Suppose that $f(u) \in \{1, 1'\}$ and $f(v) = 0$. Then from the child node t', we relax the constraint of $f(v) = 0'$. In the solution of t' we do not need to dominate v. Thus, $c[t, f] \le c[t', f_{v\to 0'}]$.
   On the contrary, suppose that $c[t', f_{v\to 0'}]$ is the minimum weight dominating set for the graph $G_{t'}$. Now in G_t, $E(G_t) = \{\{u, v\}\} \cup E(G_{t'})$. In $G_t, G_{t'}$, either $f(u) = 1$, or $f(u) = 1'$. Thus, $c[t', f_{v\to 0'}] \le c[t, f]$. Therefore, $c[t, f] = c[t', f_{v\to 0'}]$.
   Proof of other cases can be proved similarly.

4. **Forget node.** Let t be a forget node with a child t'. Then for some $w \in X_{t'}$, $X_t = X_{t'} \setminus \{w\}$. Every coloring of t' that maps w to 1 or 0 is a valid coloring. Thus, we obtain the following recursion.

$$c[t, f] = \min\{c[t', f_{w\to 1}], c[t', f_{w\to 0}]\}$$

5. **Join node.** Let t be a join node with children t_1 and t_2. Then we know that $X_t = X_{t_1} = X_{t_2}$. We say that colorings f_1 of X_{t_1} or f_2 of X_{t_2} are consistent with a coloring f of X_t if for every $v \in X_t$ the following conditions hold;
   (a) $f(v) = 1$ if and only if $f_1(v) = f_2(v) = 1$.
   (b) $f(v) = 1'$ if and only if $f_1(v) = f_2(v) = 1'$.
   (c) $f(v) = 2$ if and only if $f_1(v) = f_2(v) = 2$
   (d) $f(v) = 0$ if and only if $(f_1(v), f_2(v)) \in \{(0', 0), (0, 0')\}$
   (e) $f(v) = 0'$ if and only if $f_1(v) = f_2(v) = 0'$
   Observe that D is a compatible set for t and f if and only if D must be a compatible set for t_1 and f_1, and t_2 and f_2. Let D_1 in G_{t_1} and D_2 in G_{t_2}. Since $D_1 \subseteq D$ and $D_2 \subseteq D$, $D \cap X_t = D_1 \cap X_{t_1} = D_2 \cap X_{t_2} = f^{-1}(1) \cup f^{-1}(1')$. Thus, $|D| = |D_1| + |D_2| - |f^{-1}(1) \cup f^{-1}(1')|$. Hence, we obtain the following recursion.

$$c[t, f] = \min_{f_1, f_2}\{c[t_1, f_1] + c[t_2, f_2] - |f^{-1}(1)| - |f^{-1}(1')|\}$$

Running time. The time need to compute f at a leaf node is constant. For introduce vertex/edge node or forget node, $c[t, f]$ can be computed in $O(5^k)$ time. For join node, for every $v \in X_t$, we need to check for a pair f_1, f_2 that is consistent with f. We have $6^{|X_t|}$ triples of colorings. Thus one can solve the semitotal dominating set problem in G in time $O(6^l l^{O(1)} n)$.

Recall that in Section 4.1, we have established PSD is W[2]-hard on d-degenerate for arbitrary d. It is natural to ask for a fixed d. Interesting from Section5.2, d-degenerate for fixed d PSD is FPT.

Conclusion and Directions for further research. We analyzed the complexity of PSD parameterized by solution size on graphs with girth three, four, or at least five. We also analyzed the complexity of the problem when parameterized by cliquewidth, and treewidth.

It is an interesting direction to find an explicit algorithm for PSD parameterized by cliquewidth of the input graph.

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